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A NOTE ON CONSISTENT QUASI-GEOSTROPHIC BOUNDARY CONDITIONS IN PARTIALLY OPEN, SIMPLY AND MULTIPLY CONNECTED DOMAINS

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ABSTRACT

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Mass conservation is used to elucidate differences between initial boundary value problems for the quasi-geostrophic vorticity equation in regions that are fully enclosed versus regions that are partially open. In fully closed domains, mass conservation is explicitly imposed as a consistency constraint on the dynamical evolution of the flow. Partially or fully open domains do not require an explicit mass conservation constraint. In these cases, the mass conservation balance is used as a diagnostic to describe the interaction between open boundary conditions and the inflow/outflow vorticity implied by the quasi-geostrophic equation. The new formalism is carried over to multiply connected, partially open domains, where explicit circulation integral constraints are required around island boundaries. Implications concerning coastal applications of quasi-geostrophic numerical models are outlined.

1. INTRODUCTION

The relevance of the quasi-geostrophic (QG) approximation to the study of open ocean, non-linear, mesoscale flows has been thoroughly documented (e.g. Bretherton and Karweit, 1975; McWilliams and Flierl, 1976; Robinson et al., 1986; Pinardi and Robinson, 1987). Current efforts are directed toward extending QG applications into regions influenced by the presence

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of coasts and islands. McWilliams (1977) has derived consistent quasi-geostrophic boundary conditions for simply connected and multiply connected, closed domains. In this note, we extend the work of McWilliams (1977) to derive consistent QG boundary conditions for simply and multiply connected domains that are partially open at the external boundaries. The fully open simply connected domain case has been dealt with by Haidvogel et al. (1980) and Miller et al. (1983).

The point of this note hinges on the implications of a mass conservation statement for the familiar QG equations. These equations can be derived by a Rossby number expansion of the scaled primitive equations. For example, we formally expand the velocity field $\mathbf{v} = (u, v)$ in the Rossby number parameter ϵ as

$$\mathbf{v} = \mathbf{v}_0 + \epsilon \mathbf{v}_1 + \epsilon^2 \mathbf{v}_2 + \cdots$$

The remaining primitive equation field variables are treated in kind. Recall that the $O(\epsilon^0)$ geostrophic equations are degenerate and that the prognostic QG equations for the $O(\epsilon^0)$ quantities are taken from the $O(\epsilon^1)$ balance in the expansion. We impose an impermeable wall boundary condition for the primitive equations as $\mathbf{v} \cdot \mathbf{n} = 0$. The Rossby number expansion then yields $\mathbf{v}_0 \cdot \mathbf{n} = 0$, $\epsilon \mathbf{v}_1 \cdot \mathbf{n} = 0$, $\epsilon^2 \mathbf{v}_2 \cdot \mathbf{n} = 0$, etc; the normal velocity component must vanish to all orders in ϵ. However, the *consistent* wall boundary condition specification for the dynamical balance at a *specific* order in the Rossby number expansion, say $O(\epsilon^i)$, does *not* correspond to $\mathbf{v}_i \cdot \mathbf{n} = 0$. Instead, mass conservation to $O(\epsilon^i)$ in a closed domain only requires pointwise impermeability at the wall to $O(\epsilon^{i-1})$, that is, $\mathbf{v}_{i-1} \cdot \mathbf{n} = 0$. A weaker integral constraint is imposed on $\mathbf{v}_i \cdot \mathbf{n}$ over the wall.

In the case of the QG equations, we will review a result from McWilliams (1977) that states that the weaker condition on the first-order velocity field along the wall is $\int \mathbf{v}_1 \cdot \mathbf{n} \, ds = 0$, which in addition to $\mathbf{v}_0 \cdot \mathbf{n} = 0$ properly poses the QG boundary value problem in a fully closed domain. The stronger pointwise constraint on $\mathbf{v}_1 \cdot \mathbf{n}$ is imposed in the closure of the next higher order dynamical balance at $O(\epsilon^2)$. The demonstration here will be based on the implications of a mass conservation statement in quasi-geostrophy. This approach will be extended to show that in the case of a partially enclosed domain in QG, no constraint on $\mathbf{v}_1 \cdot \mathbf{n}$ is required. Miller et al. (1983) have already shown for the fully open boundary case that no explicit constraint is involved.

We begin now with a brief description of the QG formalism used throughout this note. It is well known that the Rossby number expansion yields a horizontally non-divergent relation for mass conservation at $O(\epsilon^0)$. Mass conservation to $O(\epsilon^1)$ is given by

$$-\nabla \cdot \mathbf{v}_1 = w_{1z} \tag{1}$$

We can expand the left-hand side of eqn. (1) using the momentum equations on a β plane as

$$-\nabla \cdot \mathbf{v}_1 = (\nabla^2 p_0)_t + \mathbf{v}_0 \cdot \nabla(\nabla^2 p_0) + \beta p_{0x} - F_h(\nabla^2 p_0) \tag{2}$$

where integer subscripts indicate the order in ϵ of a given variable; letter subscripts indicate differentiation; $\mathbf{v}_0 = (-p_{0y}, p_{0x})$ and p_0 is the streamfunction or geostrophic pressure field; β is the variation of the Coriolis parameter expanded about the central latitude of the domain; and F_h is the horizontal part of a scale-selective eddy diffusion operator acting on the relative vorticity, $R_0 = \nabla^2 p_0$. For now, we can assume that F_h is a linear differential operator of unspecified order and parametric range. For example, McWilliams (1977) uses $F_h = A_h \nabla^2$ and Robinson and Walstad (1987) use a Shapiro filter of arbitrary order in the QG numerical model.

The right-hand side of eqn. (1) can be written

$$w_{1z} = -(\sigma p_{0z})_{zt} - \mathbf{v}_0 \cdot \nabla(\sigma p_{0z})_z + F_v[(\sigma p_{0z})_z] \tag{3}$$

from the $O(\epsilon^1)$ terms in the z derivative of the thermodynamic equation. Here we use $\sigma = f_0^2/N^2(z)$, a measure of the local stratification where: $N^2(z)$ is a profile of the square of the Brunt–Vaisala frequency; and f_0 is the constant part of the Coriolis parameter. Note that the vertical part of the diffusion operator F_v acts on thermal vorticity, $T_0 = (\sigma p_{0z})_z$.

Naturally, combining eqns. (1) and (2), and using eqn. (3), we recover the familiar QG equation

$$\begin{aligned}&(\nabla^2 p_0)_t + \mathbf{v}_0 \cdot \nabla(\nabla^2 p_0) + \beta v_0 \\ &\quad = -(\sigma p_{0z})_{zt} - \mathbf{v}_0 \cdot \nabla(\sigma p_{0z})_z + F_h(\nabla^2 p_0) + F_v[(\sigma p_{0z})_z]\end{aligned} \tag{4}$$

We will use eqns. (1–3) in section 2 to derive consistent QG boundary conditions for simply connected, fully enclosed and partially or fully open domains. A diagnostic relation that monitors the effects of different forms of the QG boundary conditions will be described in detail. In section 3, we extend the analysis to multiply connected, partially open domains.

2. THE SIMPLY CONNECTED CASE

In this section we start with a reproduction of the results of McWilliams (1977) for closed, simply connected domains (Fig. 1a), so that we can compare them with the partially open domain case to be developed. A standard definition of a simply connected region Ω requires that any closed curve lying in Ω can be shrunk continuously to a point without leaving Ω.

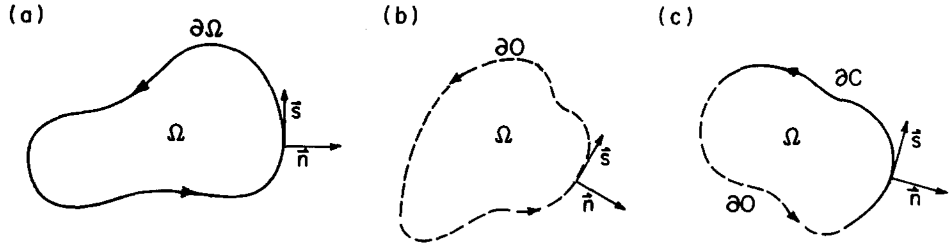

Fig. 1. Schematic boundary configurations for simply connected domains: (a) fully closed; (b) fully open; (c) partially open, $\partial\Omega = \partial C + \partial O$. Unit tangent (**s**) and normal (**n**) vectors are indicated; and the direction of integration along the boundary $\partial\Omega$, is also shown.

2.1. Closed domains

The *kinematic* QG boundary condition on horizontal walls of Ω is $\mathbf{v}_0 \cdot \mathbf{n} = 0$, which implies.

$$\nabla p_0 \cdot \mathbf{s} = 0 \quad \text{on } \partial\Omega \tag{5a}$$

where $(\mathbf{s}, \mathbf{n})$ are the unit tangent and normal vectors at the wall, $\partial\Omega$, bounding Ω (Fig. 1a). The *dynamical* QG boundary conditions, which involve the tangential velocity component $\nabla p_0 \cdot \mathbf{n}$, depend upon the physical situation of interest and the mathematical considerations appropriate to the form of the operator F_h. Here we avoid this interesting but complex problem and concentrate on the implications of kinematics.

The boundary conditions in the vertical assume a rigid lid at $z = 0$, and restricted bottom topographic variations $\nabla h = O(\epsilon)$ about a mean depth $z = -H$. The vertical boundary conditions are

$$w_1 = \boldsymbol{\kappa} \cdot \nabla \times \boldsymbol{\tau} \quad \text{at } z = 0 \tag{5b}$$

$$w_1 = \mathbf{v}_0 \cdot \nabla h \quad \text{at } z = -H \tag{5c}$$

where: $\boldsymbol{\kappa}$ is the unit vector perpendicular to the x, y plane; and $\boldsymbol{\tau} = (\tau_x, \tau_y)$ is a surface shear stress that is generally a function of the atmospheric wind velocity $\mathbf{v}_{air}$ and the ocean surface velocity $\mathbf{v}_{ocean}$. Formally, we write $\boldsymbol{\tau} = f(\mathbf{v}_{air}, \mathbf{v}_{ocean})$.

The boundary conditions (eqn. (5a)) specify that, for a baroclinic fluid, p_0 along the wall is a function of the vertical coordinates and the time only, say $g(z, t)$, and we require an explicit *consistency condition* to determine $g(z, t)$. The consistency condition derives from a volume integral of eqn. (1) over the entire domain, e.g.

$$\int_{-H}^{z} \mathrm{d}z \int\int w_{1z} \,\mathrm{d}\Omega = -\int_{-H}^{z} \mathrm{d}z \int\int \nabla \cdot \mathbf{v}_1 \,\mathrm{d}\Omega$$

where $d\Omega = dx\, dy$ is an infinitesimal surface area element. For a closed domain, the consistency condition allows mass redistribution within the domain, but must not permit net mass flow through the boundaries. Thus, using the divergence theorem, mass conservation is enforced by

$$\int_{\partial\Omega} \mathbf{v}_1 \cdot \mathbf{n}\, ds = 0 \tag{6}$$

which only constrains the *integral* of the $O(\epsilon)$ normal velocity component. That is, eqn. (6) does not require $\mathbf{v}_1 \cdot \mathbf{n}$ to vanish point by point along $\partial\Omega$; only the net flux must be zero, as we described in the introduction.

From eqn. (6), the consistency condition is then

$$\int_{-H}^{z} dz \int\int w_{1z}\, d\Omega = 0 \tag{7}$$

Integrating vertically and using eqn. (5c), eqn. (7) is identically zero at $z = -H$ (by eqn. (5a and c)). At $z = 0$, we must also require that

$$\int\int \boldsymbol{\kappa} \cdot \nabla \times \boldsymbol{\tau}\, d\Omega = \int\int \boldsymbol{\kappa} \cdot \nabla \times \left[f(\mathbf{v}_{\text{air}}, \mathbf{v}_{\text{ocean}}) \right] d\Omega = 0 \tag{8}$$

Then, using eqn. (3) we can rewrite eqn. (7) as

$$\int\int w_1(z)\, d\Omega = \int\int \left[-\sigma p_{0zt} + F_v(\sigma p_{0z}) \right] d\Omega = 0 \tag{9}$$

for every z in the interior of the fluid. We use eqns. (8) and (9) to determine $g(z, t)$ along the walls. Holland (1978) and McWilliams et al. (1978) find $g(z, t)$ in a numerical model using eqn. (9).

The surface boundary condition approach that we have introduced here requires the domain average curl of the total surface stress $\boldsymbol{\tau}$ to vanish (eqn. (8)). The contribution from $\mathbf{v}_{\text{ocean}}$ in $\boldsymbol{\tau}$ (i.e. the stress exerted on the atmosphere by the motion of the sea surface) imposes a constraint on the circulation. One effect of a western boundary current in our formulation would be to make up the balance of the net surface stress over a closed ocean basin such that eqn. (8) is preserved.

2.2. *Fully open domains*

The QG boundary conditions at open boundaries, of simply connected domains (Fig. 1b), were given by Charney, Fjortoft and von Neumann (CFvN) (1950). The CFvN conditions require specifications of p_0 at all open boundary points, and vorticity, $q_0 = R_0 + T_0$, at inflow points on the open boundary. We write

$$p_0 = p_{0\text{spec}} \quad \text{on } \partial O \tag{10}$$

$$q_0 = q_{0\text{spec}} \quad \text{on } \partial O \text{ at inflow} \tag{11}$$

where the boundary has been renamed ∂O to indicate that it is open.

Now the mass can increase (decrease) in a domain average sense, over time, according to the inflow and outflow prescribed by eqn. (10), and according to the vorticity at inflow prescribed by eqn. (11). In addition, the solution of the interior QG equation (eqn. (4)) modulates the effect of vorticity at inflow, and predicts vorticity at outflow. We have already shown that the statement of QG mass balance (eqn. (1)) is contained in the QG equation (eqn. (4)) in a pointwise sense. In an open domain, the average mass balance is implicitly maintained by satisfying the QG vorticity equation in the interior and predicting vorticity at outflow boundaries of the domain. Integrating the QG equation (eqn. (4)) over the area of the open domain, we obtain the average mass balance relation for all z

$$\int\int w_{1z}\, \mathrm{d}\Omega = -\int_{\partial O} \mathbf{v}_1 \cdot \mathbf{n}\, \mathrm{d}s \tag{12}$$

We will expand the terms in eqn. (12) when we consider partially open domains in the next section. In terms of a numerical model, one provides the CFvN conditions of eqns. (10) and (11) to start a given timestep. The QG model calculates new distributions of vorticity and streamfunction such that eqn. (12) is valid at each level. A new, mass-balance preserving vorticity is produced at outflow boundary grid points (Miller et al., 1983).

In contrast to the closed domain case of section 2.1, a consistency condition will over determine the open domain problem. In the closed domain case, the boundaries are impermeable to ν_0, and the QG equations cannot modulate inflow vorticity or predict vorticity at outflow to conserve mass, so a constraint (eqn. (9)) must be imposed.

2.3. Partially open domains

We now consider a domain boundary that is partly wall, and partly open, denoted by $\partial\Omega = \partial O + \partial C$ (see Fig. 1c). As in the fully open case (section 2.2) the QG equations can adjust vorticity at outlow such that eqn. (12) is satisfied along ∂O, and we do not require an explicit consistency constraint. But, as in the closed domain case (section 2.1), the kinematic side boundary condition (eqn. (5a)) applies at ∂C. The $g(z, t)$ variation in p_0 along ∂C can be determined by connecting the specification (eqn. (10)) smoothly from an adjacent point on ∂O to the end-point of ∂C and imposing eqn. (5a).

The mass balance relation now takes the form

$$\int\int w_{1z}\, \mathrm{d}\Omega = -\int_{\partial O + \partial C} \mathbf{v}_1 \cdot \mathbf{n}\, \mathrm{d}s \tag{13}$$

for every z. Again, eqn. (13) is not an explicit constraint that we impose in a partially open domain. It is implicitly satisfied by the QG equations in determining the vorticity at outflow along ∂O, as in the fully open domain case.

We can use eqn. (13) in a diagnostic way to examine the effects of a particular choice of $g(z, t)$. Expanding $\mathbf{v}_1$ we write

$$\mathbf{v}_1 = \boldsymbol{\kappa} \times \nabla p_1 - (\beta y)\boldsymbol{\kappa} \times \nabla p_0 - (\nabla p_0)_t - \mathbf{v}_0 \cdot \nabla(\nabla p_0) + F_h(\nabla p_0)$$

where $\mathbf{v}_0 = \boldsymbol{\kappa} \times \nabla p_0$ has been used. Considering only the flow normal to the boundary we have

$$\mathbf{v}_1 \cdot \mathbf{n} = -\nabla p_1 \cdot \mathbf{s} + \nabla p_0 \cdot \mathbf{s}(\beta y) - (\nabla p_0 \cdot \mathbf{n})_t + \nabla p_0 \cdot \mathbf{s}(p_{0nn}) - \left(\tfrac{1}{2}(\nabla p_0 \cdot \mathbf{n})^2\right)_s + F_h(\nabla p_0 \cdot \mathbf{n})$$

so eqn. (13) can be rewritten using eqn. (3) as

$$-\iint (\sigma p_{0z})_{zt}\, \mathrm{d}\Omega - \int_{\partial O} \mathbf{v}_0 \cdot \mathbf{n}(\sigma p_{0z})_z\, \mathrm{d}s + \iint F_v[(\sigma p_{0z})_z]\, \mathrm{d}\Omega$$
$$= \int_{\partial C + \partial O} \Big\{ \nabla p_1 \cdot \mathbf{s} - \nabla p_0 \cdot \mathbf{s}(\beta y) - \nabla p_0 \cdot \mathbf{s}(p_{0nn})$$
$$+ (\nabla p_0 \cdot \mathbf{n})_t + \left(\tfrac{1}{2}(\nabla p_0 \cdot \mathbf{n})^2\right)_s \Big\}\, \mathrm{d}s - \iint F_h(\nabla^2 p_0)\, \mathrm{d}\Omega$$

Since p_1 must be single valued on $\partial\Omega$, the first term on the right-hand side vanishes. Similarly, the last term of the line integral vanishes because $\partial O + \partial C$ is a closed circuit. We can finally write

$$\iint (\sigma p_{0z})_{zt}\, \mathrm{d}\Omega - \iint F_h(\nabla^2 p_0)\, \mathrm{d}\Omega - \iint F_v[(\sigma p_{0z})_z]\, \mathrm{d}\Omega$$
$$= \int_{\partial O} [\nabla p_0 \cdot \mathbf{s}(\beta y) + \nabla p_0 \cdot \mathbf{s}(p_{0nn}) - (\nabla p_0 \cdot \mathbf{n})_t]\, \mathrm{d}s - \int_{\partial O} \mathbf{v}_0 \cdot \mathbf{n}(\sigma p_{0z})_z\, \mathrm{d}s$$
$$- \int_{\partial C} (\nabla p_0 \cdot \mathbf{n})_t\, \mathrm{d}s \tag{14}$$

We separate the contributions to the balance in eqn. (14) into three parts as follows

(a) $\displaystyle \iint (\sigma p_{0z})_{zt}\, \mathrm{d}\Omega - \iint F_h(\nabla^2 p_0)\, \mathrm{d}\Omega - \iint F_v[(\sigma p_{0z})_z]\, \mathrm{d}\Omega$

Part (a) is the combined effects of dissipation (second and third terms) with thermal vorticity time rate of change, over the whole domain

(b) $\displaystyle -\int_{\partial O} \mathbf{v}_0 \cdot \mathbf{n}(\sigma p_{0z})_z\, \mathrm{d}s$

$$+ \int_{\partial O} \{\nabla p_0 \cdot \mathbf{s}(\beta y) + \nabla p_0 \cdot \mathbf{s}(p_{0nn}) - (\nabla p_0 \cdot \mathbf{n})_t\}\, \mathrm{d}s$$

Part (b) represents the inflow(outflow) of thermal, planetary, and relative vorticity at the open boundary ∂O, as well as the time variation of the tangential flow along ∂O, which can modulate prescribed vorticity at inflow and affect the prediction of vorticity at outflow

$$\text{(c)} \quad -\int_{\partial C} (\nabla p_0 \cdot \mathbf{n})_t \, \mathrm{d}s$$

Part (c) is the time variation in the tangential velocity along ∂C. Notice that we control the contribution of part (c) through the specification of the dynamic boundary condition.

In the free-slip case, the along the wall component of relative vorticity $R_0 = p_{0ss} + p_{0nn}$ is identically zero. However, the specification of $g(z, t)$ at ∂C, can induce changes in thermal vorticity, T_0, which changes the strength of the flow parallel to ∂C (part (c)). In the no slip case the (c) part of eqn. (14) vanishes. This time, the density surface variations on ∂C are only balanced by the dissipation terms in part (a), and the inflow(outflow) terms in part (b). In general, we can expect a non-zero value of p_{nn} at the wall, which will parameterize the effects of an unresolved boundary layer. For example, in the classical case of Rossby wave reflection from a western boundary, the value of the vorticity at the wall is not zero (see Pedlosky, 1987).

3. THE MULTIPLY CONNECTED CASE

In this section we introduce the presence of islands to the fully closed and partially open domains of interest (Fig. 2). We will see that in the multiply

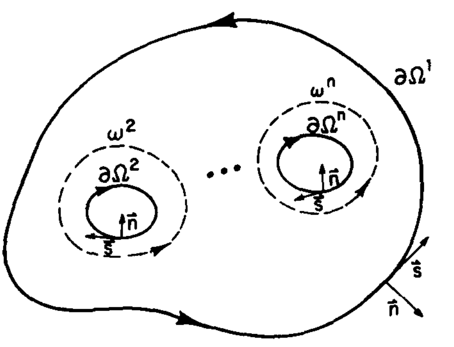

Fig. 2. A multiply connected domain with islands. The fully closed case is depicted here, where $\partial\Omega^1$ is a solid wall. In the partially open case, $\partial\Omega^1 = \partial C + \partial O$, as in Fig. 1c. Unit tangent and normal vectors (**s**, **n**)are indicated along island and domain boundaries, $\partial\Omega^i$, $i = 1, \ldots, n$. The directions of integration are shown for these boundaries, as well as for the island circuits, ω^i, $i = 2, \ldots, n$.

connected case, a circulation integral is required to fix the transport between solid boundaries within the region. We define an n-multiply connected region Ω^1, with boundary $\partial\Omega^1$, such that it is possible to draw $n-1$ island boundaries; $\partial\Omega^2, \ldots, \partial\Omega^n$, completely within $\partial\Omega^1$.

3.1. Closed domain containing islands

We begin with a digression to simple connectivity to illuminate the effect of a circulation integral constraint. In the simply connected closed boundary case already discussed (section 2.1), the mass conservation integral (eqn. (6)) trivially implies a conservation equation for the circulation as well (Pedlosky, 1987). Let the velocity circulation Γ be written as

$$\Gamma = \iint \nabla^2 p_0 \, \mathrm{d}\Omega = \int_{\partial\Omega} (\nabla p_0 \cdot \mathbf{n}) \, \mathrm{d}s$$

then if we impose eqn. (6), at every z, we obtain

$$\int_{\partial\Omega} (\nabla p_0 \cdot \mathbf{n})_t \, \mathrm{d}s = \iint F_h(\nabla^2 p_0) \, \mathrm{d}\Omega \tag{15}$$

So mass conservation (eqn. (6)) implies that the time rate of change of the velocity circulation Γ_t is balanced by the average dissipation in the basin. Furthermore, since eqns. (6) and (9) are equivalent, we know that the conditions (15) and (9) are also equivalent in the simply connected case. However, in the multiply connected closed domain case to be discussed in this section, condition (15) must be imposed explicitly on the island circuits; and condition (9) is imposed separately and explicitly as well. We will restate eqn. (15) as condition (17) for the multiply connected case presented in the following.

Here again, we review the results of McWilliams (1977) for comparison with the results of the next section. The kinematic condition (eqn. (5a)) is always required. This specifies that the streamfunction along each of the solid boundaries, $\partial\Omega^i$, $i = 1, \ldots, n$, is written as $g_i(z, t)$. To determine this variation we again enforce the mass conservation relation

$$\iint w_1 \, \mathrm{d}\Omega = 0 \tag{9}$$

at every interior z in the fluid. Furthermore, we impose a transport circulation integral, which can be defined for the volume (Fig. 2) from the island circuits, ω^i, to the island boundaries, $\partial\Omega^i$; $i = 2, \ldots, n$, and over the entire depth

$$\int_{-H}^{0} \mathrm{d}z \iint \left[(\nabla^2 p_0)_t + \mathbf{v}_0 \cdot \nabla(\nabla^2 p_0) + \beta p_{0x} - F_h(\nabla^2 p_0) \right] \mathrm{d}A^i$$

$$= \iint w_1(0)\, \mathrm{d}A^i - \iint w_1(-H)\, \mathrm{d}A^i$$

where $\mathrm{d}A^i$ is an infinitesimal horizontal area element between ω^i and $\partial\Omega^i$. Evaluating the right-hand side by using eqn. (5b and c)

$$0 = -\int_{-H}^{0} \mathrm{d}z \iint F_h(\nabla^2 p_0)\, \mathrm{d}A^i$$

$$+ \int_{-H}^{0} \mathrm{d}z \int_{\omega^i} \left\{ (\nabla p_0 \cdot \mathbf{n})_t + \nabla p_0 \cdot \mathbf{s}(p_{0nn}) + \nabla p_0 \cdot \mathbf{s}(\beta y) \right\} \mathrm{d}s$$

$$+ \int_{\omega^i} \left\{ \mathbf{v}_0 \cdot \mathbf{n}[h(x, y)] - \boldsymbol{\tau} \cdot \mathbf{s} \right\} \mathrm{d}s \tag{16}$$

Finally, we modify condition (15) to specify the circulation of the relative velocity at all z in the fluid interior, about each island boundary, $\partial\Omega^i$, as

$$0 = \int_{\partial\Omega^i} \left[(\nabla p_0 \cdot \mathbf{n})_t - F_h(\nabla p_0 \cdot \mathbf{n}) \right] \mathrm{d}s \tag{17}$$

Using the combination of eqns. (9), (16) and (17), we can determine the unknown functions $g_i(z, t)$ along the rigid walls of the domain.

3.2. *Fully or partially open domain with islands*

Finally, we consider the analogues to an island archipelago, or an island in the vicinity of a mainland coast. This geometry corresponds to changing Fig. 2 such that $\partial\Omega^1 = \partial O + \partial C$, as shown in Fig. 1c. As always, condition (5a) is imposed at solid walls (∂C and $\partial\Omega^i$, $i = 2, \ldots, n$). The transport circulation (eqn. (16)) around the islands, and the circulation of the relative velocity in each interior level (eqn. (17)) about an island must be specified as well to determine the functions of $g_i(z, t)$ at the island walls. However, as seen before in the simply connected domain case (section 2.3), we are not required to enforce a global mass balance explicitly, since $\partial\Omega^1$ is open. For example, if a domain contains two islands, and we model the stratification by two levels in the fluid, then four unknown functions of time remain to be specified; one for each island at each level. We apply eqn. (16) around each of the islands (two conditions), and eqn. (17) at one of the interior levels for each island (two more conditions) to close the problem.

4. SUMMARY AND DISCUSSION

We have presented consistent quasi-geostrophic boundary conditions for partially open, simply and multiply connected, regions of the ocean. We pose the problem of reflecting Rossby waves as a motivation for the study of quasi-geostrophic dynamics in the vicinity of rigid boundaries, such as islands and coastlines (see LeBlond and Mysak, 1978; Pedlosky, 1987). The wavenumber dependence of the Rossby wave reflection process controls the transmission of energy back into the ocean interior, or the trapping of this energy near the boundary (e.g. Pedlosky, 1965; Rhines, 1969; Flierl, 1977). Our work establishes a means for determining the necessary boundary conditions for a numerical study of this process.

We have reviewed the consistent quasi-geostrophic boundary conditions of McWilliams (1977) for fully closed domains. These conditions have been compared with the partially open domain conditions developed here. The requirement for no *geostrophic* flow through solid boundaries (condition (5a)) is common to both open and closed geometries. In a baroclinic ocean, the streamfunction is then determined up to a function $g(z, t)$ along the walls. In the case of a fully closed domain, a mass conservation integral constraint (eqn. (9)) must be explicitly imposed along the boundary circuit to fix $g(z, t)$. In the partially open domain, $g(z, t)$ can be imposed instead by the smooth connection from the interior solution at the open boundary and next to the wall, to condition (5a) along the wall. The QG equation controls the evolution of the flow field, modulates prescribed vorticity at inflow, and predicts vorticity at outflow, such that mass balance is implicitly maintained. When the domain of interest is multiply connected, explicit circulation integral constraints must be imposed to ensure that there is no net mass flux across island boundaries.

In partially open domains, the $g(z, t)$ variation along the wall can be determined by an extrapolation or matching condition from adjacent open boundary points. We note that some control over the quasi-geostrophic phenomena admitted to a particular numerical study is possible through consistent manipulation of the matching condition between ∂O and ∂C. Clearly, the form of the dynamic boundary condition, and the dissipation parameterization, will also affect the QG flow near the solid boundary portion of a partially open domain. These effects can be quantified by evaluating the terms in eqn. (14).

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